Aspects of the q–deformed Fuzzy Sphere[1]

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Abstract

These notes are a short review of the q–deformed fuzzy sphere $S^2_{q,N}$, which is a "finite" noncommutative 2–sphere covariant under the quantum group $U_q(su(2))$. We discuss its real structure, differential calculus and integration for both real q and q a phase, and show how actions for Yang–Mills and Chern–Simons–like gauge theories arise naturally. It is related to D-branes on the $SU(2)_k$ WZW model for $q = \exp(\frac{i\pi}{k+2})$.

1 Introduction

$S^2_{q,N}$ is a q–deformed version of the "ordinary" fuzzy sphere S^2_N [3]. It is covariant under the standard Drinfeld–Jimbo quantum group $U_q(su(2))$, and can be defined for both $q \in \mathbb{R}$ and $|q| = 1$. The algebra of functions on $S^2_{q,N}$ is isomorphic to the matrix algebra $Mat(N+1, \mathbb{C})$, but it carries additional structure which distinguishes it from S^2_N, related to its rotation symmetry under $U_q(su(2))$. For real q, we recover precisely the "discrete series" of Podles spheres [4]. We describe its structure in general, including a covariant differential calculus and integration, and show how actions of Yang–Mills and Chern–Simons type arise naturally on this space. A much more detailed study of $S^2_{q,N}$ has been given by in [1]. These considerations were motivated mainly by the work [2] of Alekseev, Recknagel and Schomerus, who study the boundary conformal field theory describing spherical D–branes in the $SU(2)$ WZW model at level k. These authors extract an "effective" algebra of functions on the D–branes from the OPE of the boundary vertex operators. This algebra is twist–equivalent [1] to the space of functions on $S^2_{q,N}$, if q is related to the level k by the formula

$$q = \exp(\frac{i\pi}{k+2}). \tag{1.1}$$

2 The space S^2_q

First, recall that an algebra $\mathcal{A}$ is a $U_q(su(2))$–module algebra if there exists a map

$$\begin{aligned} U_q(su(2)) \times \mathcal{A} &\rightarrow \mathcal{A}, \\ (u, a) &\mapsto u \triangleright a \end{aligned} \tag{2.1}$$

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which satisfies $u \triangleright (ab) = (u_{(1)} \triangleright a)(u_{(2)} \triangleright b)$ for $a, b \in \mathcal{A}$. Here $\Delta(u) = u_{(1)} \otimes u_{(2)}$ is the Sweedler notation for the coproduct of $u \in U_q(su(2))$.

A particularly simple way to define the q–deformed fuzzy sphere is as follows: Consider the spin $\frac{N}{2}$ representation of $U_q(su(2))$,

$$\rho : U_q(su(2)) \to Mat(N+1, \mathbb{C}), \tag{2.2}$$

which acts on $\mathbb{C}^{N+1}$. With this in mind, it is natural to consider the simple matrix algebra $Mat(N+1, \mathbb{C})$ as a $U_q(su(2))$–module algebra, by $u \triangleright M = \rho(u_1) M \rho(S u_2)$. This defines $S^2_{q,N}$. It is easy to see that under this action of $U_q(su(2))$, it decomposes into the irreducible representations

$$S^2_{q,N} := Mat(N+1, \mathbb{C}) = (1) \oplus (3) \oplus ... \oplus (2N+1) \tag{2.3}$$

(if q is a root of unity (1.1), this holds provided $N \leq k/2$, which we will assume here). Let $\{x_i\}_{i=+,-,0}$ be the weight basis of the spin 1 components, so that $u \triangleright x_i = x_j \pi^j_i(u)$ for $u \in U_q(su(2))$. One can then show that they satisfy the relations

$$\begin{aligned} \varepsilon_k^{ij} x_i x_j &= \Lambda_N x_k, \\ g^{ij} x_i x_j &= R^2. \end{aligned} \tag{2.4}$$

Here

$$\Lambda_N = R \, \frac{[2]_{q^{N+1}}}{\sqrt{[N]_q [N+2]_q}}, \tag{2.5}$$

$[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}$, and ε_k^{ij} and g^{ij} are the q–deformed invariant tensors. For example, $\varepsilon_3^{33} = q^{-1} - q$, and $g^{1-1} = -q^{-1}$, $g^{00} = 1$, $g^{-11} = -q$. In [1], these relations were derived using a Jordan–Wigner construction. For $q = 1$, the relations of S^2_N are recovered, and for real $q \neq 1$ we obtain $\varepsilon_k^{ij} x_i x_j = R\,(q - q^{-1})\, x_k$ in the limit $N \to \infty$.

Real structure. In order to define a *real* noncommutative space, one must specify a star structure on the algebra of functions. Since $S^2_{q,N}$ should decompose into unitary representations of $U_q(su(2))$, we restrict to the cases $q \in \mathbb{R}$ and $|q| = 1$. In either case, the star structure on $S^2_{q,N} = Mat(N+1, \mathbb{R})$ is defined to be the usual hermitean adjoint of matrices. In terms of the generators x_i, this becomes

$$x_i^* = g^{ij} x_j, \qquad \text{if } q \in \mathbb{R} \tag{2.6}$$

and

$$x_i^* = -\omega x_i \omega^{-1} = x_j \rho(L^{-j}{}_k) q^{-2} g^{ki}, \qquad \text{if } |q| = 1. \tag{2.7}$$

Here $\omega \in \hat{U}_q(su(2))$ generates the quantum Weyl reflection [5],

$$\Delta(\omega) = \mathcal{R}^{-1} \omega \otimes \omega, \qquad \omega^2 = v\epsilon, \qquad v = S\mathcal{R}_2 \mathcal{R}_1 q^{-H}, \tag{2.8}$$

and $L^{-i}_{\ j} = \pi^i_j(\mathcal{R}_1^{-1})\mathcal{R}_2^{-1} \in U_q(su(2))$. Using the map ρ (2.2), this amounts to the star structure $H^* = H, (X^\pm)^* = X^\mp$ defining the compact form $U_q(su(2))$ for both $q \in \mathbb{R}$ and $|q| = 1$. In the case $q \in \mathbb{R}$, we have recovered precisely the discrete series of Podles spheres [4].

To summarize, $S^2_{q,N}$ is same *algebra* $Mat(N+1, \mathbb{C})$ as $S^2_N \equiv S^2_{q=1,N}$, but its symmetry $U_q(su(2))$ acts on it in a way which is inequivalent to the undeformed case. It admits additional structure compatible with this symmetry, such as a differential calculus and an integral. This will be discussed next.

3 Differential calculus and integration

In order to write down Lagrangians, it is convenient to use the notion of an (exterior) differential calculus. A covariant differential calculus over $S_{q,N}$ is a graded bimodule $\Omega^* = \oplus_n \Omega^n$ over $S_{q,N}$ which is a $U_q(su(2))$–module algebra, together with an exterior derivative d which satisfies $d^2 = 0$ and the graded Leibnitz rule.

The structure of the calculus is determined by requiring covariance, and a systematic way to derive it is given in [1]. Here we will simply quote the most important features. First, the modules Ω^n turn out to be free over $S_{q,N}$ both as left and right modules[2] with $dim\ \Omega^n = (1,3,3,1)$ for $n = (0,1,2,3)$, and vanish for higher n. In particular, it is not possible to have a calculus with only "tangential" forms; this means that vector fields over $S_{q,N}$ will in general also contain "radial" components. As suggested by the dimensions, there exists a canonical map

$$*_H:\ \Omega^n \to \Omega^{3-n}, \tag{3.1}$$

which satisfies $(*_H)^2 = id$, and respects the $U_q(su(2))$– and $S_{q,N}$–module structures. It satisfies furthermore

$$\alpha(*_H\beta) = (*_H\alpha)\beta$$

for any $\alpha, \beta \in \Omega^*$. Moreover, there exists a special one–form

$$\Theta \in \Omega^1_{q,N}$$

which is a singlet under $U_q(su(2))$, and generates the calculus as follows:

$$\begin{aligned} df &= [\Theta, f], \\ d\alpha^{(1)} &= [\Theta, \alpha^{(1)}]_+ - *_H(\alpha^{(1)}), \\ d\alpha^{(2)} &= [\Theta, \alpha^{(2)}] \end{aligned} \tag{3.2}$$

for any $f \in S_{q,N}$ and $\alpha^{(i)} \in \Omega^i$. One can verify that

$$d\Theta = \Theta^2 = *_H(\Theta),$$

[2] but not as bimodules

and $[f, \Theta^3] = 0$ with $\Theta^3 \neq 0$. There also exists a star structure [1] on Ω^* for both $q \in \mathbb{R}$ and $|q| = 1$, which makes it a covariant $\star$ calculus.

Frame. The most convenient basis to work with is the "frame" generated by one–forms $\theta^a \in \Omega^1$ for $a = +, -, 0$, which satisfy

$$
\begin{aligned}
[\theta^a, f] &= 0, && (3.3)\\
\theta^a \theta^b &= -q^2\ \hat{R}^{ba}_{cd} \theta^d \theta^c, && (3.4)\\
*_H \theta^a &= -\frac{1}{q^2 [2]_{q^2}} \varepsilon^a_{bc} \theta^c \theta^b, &&\\
\theta^a \theta^b \theta^c &= -\Lambda_N^2 \frac{q^6}{R^2}\ \varepsilon^{cba} \Theta^3. && (3.5)
\end{aligned}
$$

Such θ^a exist and are essentially unique. The disadvantage is that they have a somewhat complicated transformation law

$$u \triangleright \theta^a = u_1 S u_3 \pi^a_b (S u_2)\ \theta^b.$$

One can alternatively use a basis of one–forms ξ_i which transform as a vector under $U_q(su(2))$, but then the commutation relations are more complicated [1].

Integration. The unique invariant integral of a function $f \in S^2_{q,N}$ is given by its quantum trace as an element of $Mat(N+1, \mathbb{C})$,

$$\int f := \frac{4\pi R^2}{[N+1]_q} \mathrm{Tr}_q(f) = \frac{4\pi R^2}{[N+1]_q} \mathrm{Tr}(f\ q^{-H}),$$

normalized such that $\int 1 = 4\pi R^2$. Invariance means that $\int u \triangleright f = \varepsilon(u)\ \int f$. It is useful to define also the integral of forms, by declaring Θ^3 to be the "volume form". Writing any 3–form as $\alpha^{(3)} = f\Theta^3$, we define $\int \alpha^{(3)} = \int f\Theta^3 := \int f$. Using the correct cyclic property of this integral, one can then verify Stokes theorem

$$\int d\alpha^{(2)} = \int [\Theta, \alpha^{(2)}] = 0. \tag{3.6}$$

4 Gauge Fields

Actions for gauge theories arise in a very natural way on $S^2_{q,N}$. We shall describe (abelian) gauge fields as one–forms

$$B = \sum B_a \theta^a \quad \in \Omega^1,$$

expanded in terms of the frames θ^a. They have 3 independent components, which means that B has also a radial component. However, the latter cannot be disentangled from the other components, since it is not possible to construct a covariant calulus with 2 tangential components only.

We propose that Lagrangians for gauge fields should contain *no* explicit derivatives in the B fields. The kinetic terms will then arise naturally upon a shift of the field B,

$$B = \Theta + A.$$

The only Lagrangian of order ≤ 3 in B which after this shift contains no linear terms in A is the " Chern-Simons" action

$$S_{CS} := \frac{1}{3}\int B^3 - \frac{1}{2}\int B *_H B = -\text{const} + \frac{1}{2}\int AdA + \frac{2}{3}A^3. \tag{4.1}$$

Going to order 4 in B, we define the curvature as

$$F := B^2 - *_H B = dA + A^2,$$

using (3.2). Then a "Yang–Mills" action is naturally obtained as

$$S_{YM} := \int F *_H F = \int (dA + A^2) *_H (dA + A^2). \tag{4.2}$$

These are precisely the kind of actions that have been found [2] in the string–induced low–energy effective action on D–branes in the $SU(2)_k$ WZW model, in the leading approximation.

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